

4.E: Line and Surface Integrals (Exercises)

4.1: Line Integrals

A

For Exercises 1-4, calculate $\int_C f(x, y) ds$ for the given function $f(x, y)$ and curve C .

4.1.1. $f(x, y) = xy$; $C : x = \cos t, y = \sin t, 0 \leq t \leq \pi/2$

4.1.2. $f(x, y) = \frac{x}{x^2} + 1$; $C : x = t, y = 0, 0 \leq t \leq 1$

4.1.3. $f(x, y) = 2x + y$; C : polygonal path from $(0, 0)$ to $(3, 0)$ to $(3, 2)$

4.1.4. $f(x, y) = x + y^2$; C : path from $(2, 0)$ counterclockwise along the circle $x^2 + y^2 = 4$ to the point $(-2, 0)$ and then back to $(2, 0)$ along the x -axis

4.1.5. Use a line integral to find the lateral surface area of the part of the cylinder $x^2 + y^2 = 4$ below the plane $x + 2y + z = 6$ and above the xy -plane.

For Exercises 6-11, calculate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for the given vector field $\mathbf{f}(x, y)$ and curve C .

4.1.6. $\mathbf{f}(x, y) = \mathbf{i} - \mathbf{j}$; $C : x = 3t, y = 2t, 0 \leq t \leq 1$

4.1.7. $\mathbf{f}(x, y) = y\mathbf{i} - x\mathbf{j}$; $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$

4.1.8. $\mathbf{f}(x, y) = x\mathbf{i} + y\mathbf{j}$; $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$

4.1.9. $\mathbf{f}(x, y) = (x^2 - y)\mathbf{i} + (x - y^2)\mathbf{j}$; $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$

4.1.10. $\mathbf{f}(x, y) = xy^2\mathbf{i} + xy^3\mathbf{j}$; C : the polygonal path from $(0, 0)$ to $(1, 0)$ to $(0, 1)$ to $(0, 0)$

4.1.11. $\mathbf{f}(x, y) = (x^2 + y^2)\mathbf{i}$; $C : x = 2 + \cos t, y = \sin t, 0 \leq t \leq 2\pi$

B

4.1.12. Verify that the value of the line integral in Example 4.1 is unchanged when using the parametrization of the circle C given in formulas Equation (4.8).

4.1.13. Show that if $\mathbf{f} \perp \mathbf{r}'(t)$ at each point $\mathbf{r}(t)$ along a smooth curve C , then $\int_C \mathbf{f} \cdot d\mathbf{r} = 0$.

4.1.14. Show that if \mathbf{f} points in the same direction as $\mathbf{r}'(t)$ at each point $\mathbf{r}(t)$ along a smooth curve C , then $\int_C \mathbf{f} \cdot d\mathbf{r} = \int_C \|\mathbf{f}\| ds$.

C

4.1.15. Prove that $\int_C f(x, y) ds = \int_{-C} f(x, y) ds$. (Hint: Use formulas Equation (4.9).)

4.1.16. Let C be a smooth curve with arc length L , and suppose that $\mathbf{f}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field such that $\|\mathbf{f}(x, y)\| \leq M$ for all (x, y) on C . Show that $|\int_C \mathbf{f} \cdot d\mathbf{r}| \leq ML$. (Hint: Recall that $|\int_a^b g(x) dx| \leq \int_a^b |g(x)| dx$ for Riemann integrals.)

4.1.17. Prove that the Riemann integral $\int_a^b f(x) dx$ is a special case of a line integral.

4.2: Properties of Line Integrals

A

4.2.1. Evaluate $\oint_C (x^2 + y^2) dx + 2xy dy$ for $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$.

4.2.2. Evaluate $\int_C (x^2 + y^2) dx + 2xy dy$ for $C : x = \cos t, y = \sin t, 0 \leq t \leq \pi$

4.2.3. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = y\mathbf{i} - x\mathbf{j}$? If so, find one.

4.2.4. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = x\mathbf{i} - y\mathbf{j}$? If so, find one.

4.2.5. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = xy^2\mathbf{i} + x^3y\mathbf{j}$? If so, find one.

B

4.2.6. Let $\mathbf{f}(x, y)$ and $\mathbf{g}(x, y)$ be vector fields, let a and b be constants, and let C be a curve in \mathbb{R}^2 . Show that

$$\int_C (a\mathbf{f} \pm b\mathbf{g}) \cdot d\mathbf{r} = a \int_C \mathbf{f} \cdot d\mathbf{r} \pm b \int_C \mathbf{g} \cdot d\mathbf{r}. \quad (4.E.1)$$

4.2.7. Let C be a curve whose arc length is L . Show that $\int_C 1 ds = L$.

4.2.8. Let $f(x, y)$ and $g(x, y)$ be continuously differentiable real-valued functions in a region R . Show that

$$\oint_C f \nabla g \cdot d\mathbf{r} = - \oint_C g \nabla f \cdot d\mathbf{r} \quad (4.E.2)$$

for any closed curve C in R . (Hint: Use Exercise 21 in Section 2.4.)

4.2.9. Let $\mathbf{f}(x, y) = \frac{-y}{x^2 + y^2} \mathbf{i} + \frac{x}{x^2 + y^2} \mathbf{j}$ for all $(x, y) \neq (0, 0)$, and $C : x = \cos t, y = \sin t, 0 \leq t \leq 2\pi$.

(a) Show that $\mathbf{f} = \nabla F$, for $F(x, y) = \tan^{-1}(y/x)$.

(b) Show that $\oint_C \mathbf{f} \cdot d\mathbf{r} = 2\pi$. Does this contradict Corollary 4.6? Explain.

C

4.2.10. Let $g(x)$ and $h(y)$ be differentiable functions, and let $\mathbf{f}(x, y) = h(y)\mathbf{i} + g(x)\mathbf{j}$. Can \mathbf{f} have a potential $F(x, y)$? If so, find it. You may assume that F would be smooth. (Hint: Consider the mixed partial derivatives of F .)

4.3: Green's Theorem

A

For Exercises 1-4, use Green's Theorem to evaluate the given line integral around the curve C , traversed counterclockwise.

4.3.1. $\oint_C (x^2 - y^2) dx + 2xy dy$; C is the boundary of $R = (x, y) : 0 \leq x \leq 1, 2x^2 \leq y \leq 2x$

4.3.2. $\oint_C x^2 y dx + 2xy dy$; C is the boundary of $R = (x, y) : 0 \leq x \leq 1, x^2 \leq y \leq x$

4.3.3. $\oint_C 2y dx - 3x dy$; C is the circle $x^2 + y^2 = 1$

4.3.4. $\oint_C (e^{x^2} + y^2) dx + (e^{y^2} + x^2) dy$; C is the boundary of the triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 4)$

4.3.5. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = (y^2 + 3x^2)\mathbf{i} + 2xy\mathbf{j}$? If so, find one.

4.3.6. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = (x^3 \cos(xy) + 2x \sin(xy))\mathbf{i} + x^2 y \cos(xy)\mathbf{j}$? If so, find one.

4.3.7. Is there a potential $F(x, y)$ for $\mathbf{f}(x, y) = (8xy + 3)\mathbf{i} + 4(x^2 + y)\mathbf{j}$? If so, find one.

4.3.8. Show that for any constants a, b and any closed simple curve C , $\oint_C a dx + b dy = 0$.

B

4.3.9. For the vector field \mathbf{f} as in Example 4.8, show directly that $\oint_C \mathbf{f} \cdot d\mathbf{r} = 0$, where C is the boundary of the annulus $R = (x, y) : 1/4 \leq x^2 + y^2 \leq 1$ traversed so that R is always on the left.

4.3.10. Evaluate $\oint_C e^x \sin y dx + (y^3 + e^x \cos y) dy$, where C is the boundary of the rectangle with vertices $(1, -1)$, $(1, 1)$, $(-1, 1)$ and $(-1, -1)$, traversed counterclockwise.

C

4.3.11. For a region R bounded by a simple closed curve C , show that the area A of R is

$$A = - \oint_C y dx = \oint_C x dy = \frac{1}{2} \oint_C x dy - y dx, \quad (4.E.3)$$

where C is traversed so that R is always on the left. (Hint: Use Green's Theorem and the fact that $A = \iint_R 1 dA$.)

4.4: Surface Integrals and the Divergence Theorem

A

For Exercises 1-4, use the Divergence Theorem to evaluate the surface integral $\iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$ of the given vector field $\mathbf{f}(x, y, z)$ over the surface Σ .

4.4.1. $\mathbf{f}(x, y, z) = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 9$

4.4.2. $\mathbf{f}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, Σ : boundary of the solid cube $S = (x, y, z) : 0 \leq x, y, z \leq 1$

4.4.3. $\mathbf{f}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

4.4.4. $\mathbf{f}(x, y, z) = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, $\Sigma : x^2 + y^2 + z^2 = 1$

B

4.4.5. Show that the flux of any constant vector field through any closed surface is zero.

4.4.6. Evaluate the surface integral from Exercise 2 *without* using the Divergence Theorem, i.e. using only Definition 4.3, as in Example 4.10. Note that there will be a different outward unit normal vector to each of the six faces of the cube.

4.4.7. Evaluate the surface integral $\iint_{\Sigma} \mathbf{f} \cdot d\boldsymbol{\sigma}$, where $\mathbf{f}(x, y, z) = x^2\mathbf{i} + xy\mathbf{j} + z\mathbf{k}$ and Σ is the part of the plane $6x + 3y + 2z = 6$ with $x \geq 0$, $y \geq 0$, and $z \geq 0$, with the outward unit normal \mathbf{n} pointing in the positive z direction.

4.4.8. Use a surface integral to show that the surface area of a sphere of radius r is $4\pi r^2$. (Hint: Use spherical coordinates to parametrize the sphere.)

4.4.9. Use a surface integral to show that the surface area of a right circular cone of radius R and height h is $\pi R \sqrt{h^2 + R^2}$. (Hint: Use the parametrization $x = r \cos \theta$, $y = r \sin \theta$, $z = \frac{h}{R} r$, for $0 \leq r \leq R$ and $0 \leq \theta \leq 2\pi$.)

4.4.10. The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ can be parametrized using *ellipsoidal* coordinates

$$x = a \sin \varphi \cos \theta, y = b \sin \varphi \sin \theta, z = c \cos \varphi, \text{ for } 0 \leq \theta \leq 2\pi \text{ and } 0 \leq \varphi \leq \pi. \quad (4.E.4)$$

Show that the surface area S of the ellipsoid is

$$S = \int_0^\pi \int_0^{2\pi} \pi \sin \varphi \sqrt{a^2 b^2 \cos^2 \varphi + c^2 (a^2 \sin^2 \theta + b^2 \cos^2 \theta) \sin^2 \varphi} d\theta d\varphi. \quad (4.E.5)$$

(Note: The above double integral can not be evaluated by elementary means. For specific values of a, b and c it can be evaluated using numerical methods. An alternative is to express the surface area in terms of *elliptic integrals*.)

C

4.4.11. Use Definition 4.3 to prove that the surface area S over a region R in \mathbb{R}^2 of a surface $z = f(x, y)$ is given by the formula

$$S = \iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2} dA. \quad (4.E.6)$$

(Hint: Think of the parametrization of the surface.)

4.5: Stokes' Theorem

A

For Exercises 1-3, calculate $\int_C f(x, y, z) ds$ for the given function $f(x, y, z)$ and curve C .

4.5.1. $f(x, y, z) = z$; $C : x = \cos t, y = \sin t, z = t, 0 \leq t \leq 2\pi$

4.5.2. $f(x, y, z) = \frac{x}{y} + y + 2yz$; $C : x = t^2, y = t, z = 1, 1 \leq t \leq 2$

4.5.3. $f(x, y, z) = z^2$; $C : x = t \sin t, y = t \cos t, z = \frac{2\sqrt{2}}{3}t^{3/2}, 0 \leq t \leq 1$

For Exercises 4-9, calculate $\int_C \mathbf{f} \cdot d\mathbf{r}$ for the given vector field $\mathbf{f}(x, y, z)$ and curve C .

4.5.4. $\mathbf{f}(x, y, z) = \mathbf{i} - \mathbf{j} + \mathbf{k}$; $C : x = 3t, y = 2t, z = t, 0 \leq t \leq 1$

4.5.5. $\mathbf{f}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$; $C : x = \cos t, y = \sin t, z = t, 0 \leq t \leq 2\pi$

4.5.6. $\mathbf{f}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$; $C : x = \cos t, y = \sin t, z = 2, 0 \leq t \leq 2\pi$

4.5.7. $\mathbf{f}(x, y, z) = (y - 2z)\mathbf{i} + xy\mathbf{j} + (2xz + y)\mathbf{k}$; $C : x = t, y = 2t, z = t^2 - 1, 0 \leq t \leq 1$

4.5.8. $\mathbf{f}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; C : the polygonal path from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 2, 0)$

4.5.9. $\mathbf{f}(x, y, z) = xy\mathbf{i} + (z - x)\mathbf{j} + 2yz\mathbf{k}$; C : the polygonal path from $(0, 0, 0)$ to $(1, 0, 0)$ to $(1, 2, 0)$ to $(1, 2, -2)$

For Exercises 10-13, state whether or not the vector field $\mathbf{f}(x, y, z)$ has a potential in \mathbb{R}^3 (you do not need to find the potential itself).

4.5.10. $\mathbf{f}(x, y, z) = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$

4.5.11. $\mathbf{f}(x, y, z) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ (a, b, c constant)

4.5.12. $\mathbf{f}(x, y, z) = (x + y)\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$

4.5.13. $\mathbf{f}(x, y, z) = xy\mathbf{i} - (x - yz^2)\mathbf{j} + y^2z\mathbf{k}$

B

For Exercises 14-15, verify Stokes' Theorem for the given vector field $\mathbf{f}(x, y, z)$ and surface Σ .

4.5.14. $\mathbf{f}(x, y, z) = 2y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$; $\Sigma : x^2 + y^2 + z^2 = 1, z \geq 0$

4.5.15. $\mathbf{f}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + yz\mathbf{k}$; $\Sigma : z = x^2 + y^2, z \leq 1$

4.5.16. Construct a Möbius strip from a piece of paper, then draw a line down its center (like the dotted line in Figure 4.5.3(b)). Cut the Möbius strip along that center line completely around the strip. How many surfaces does this result in? How would you describe them? Are they orientable?

4.5.17. Use Gnuplot (see Appendix C) to plot the Möbius strip parametrized as:

$$\mathbf{r}(u, v) = \cos u \left(1 + v \cos \frac{u}{2}\right) \mathbf{i} + \sin u \left(1 + v \cos \frac{u}{2}\right) \mathbf{j} + v \sin \frac{u}{2} \mathbf{k}, \quad 0 \leq u \leq 2\pi, -\frac{1}{2} \leq v \leq \frac{1}{2} \quad (4.E.7)$$

C

4.5.18. Let Σ be a closed surface and $\mathbf{f}(x, y, z)$ a smooth vector field. Show that $\iint_{\Sigma} (\text{curl } \mathbf{f}) \cdot \mathbf{n} d\sigma = 0$. (Hint: Split Σ in half.)

4.5.19. Show that Green's Theorem is a special case of Stokes' Theorem.

4.6: Gradient, Divergence, Curl, and Laplacian

A

For Exercises 1-6, find the Laplacian of the function $f(x, y, z)$ in Cartesian coordinates.

4.6.1. $f(x, y, z) = x + y + z$

4.6.2. $f(x, y, z) = x^5$

4.6.3. $f(x, y, z) = (x^2 + y^2 + z^2)^{3/2}$

4.6.4. $f(x, y, z) = e^{x+y+z}$

4.6.5. $f(x, y, z) = x^3 + y^3 + z^3$

4.6.6. $f(x, y, z) = e^{-x^2 - y^2 - z^2}$

4.6.7. Find the Laplacian of the function in Exercise 3 in spherical coordinates.

4.6.8. Find the Laplacian of the function in Exercise 6 in spherical coordinates.

4.6.9. Let $f(x, y, z) = \frac{z}{x^2 + y^2}$ in Cartesian coordinates. Find ∇f in cylindrical coordinates.

4.6.10. For $\mathbf{f}(r, \theta, z) = r\mathbf{e}_r + z\sin\theta\mathbf{e}_\theta + rz\mathbf{e}_z$ in cylindrical coordinates, find $\operatorname{div} \mathbf{f}$ and $\operatorname{curl} \mathbf{f}$.

4.6.11. For $\mathbf{f}(\rho, \theta, \varphi) = \mathbf{e}_\rho + \rho \cos\theta\mathbf{e}_\theta + \rho\mathbf{e}_\varphi$ in spherical coordinates, find $\operatorname{div} \mathbf{f}$ and $\operatorname{curl} \mathbf{f}$.

B

For Exercises 12-23, prove the given formula ($r = \|\mathbf{r}\|$ is the length of the position vector field $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$).

4.6.12. $\nabla(1/r) = -\mathbf{r}/r^3$

4.6.13. $\Delta(1/r) = 0$

4.6.14. $\nabla \cdot (\mathbf{r}/r^3) = 0$

4.6.15. $\nabla(\ln r) = \mathbf{r}/r^2$

4.6.16. $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div} \mathbf{F} + \operatorname{div} \mathbf{G}$

4.6.17. $\operatorname{curl}(\mathbf{F} + \mathbf{G}) = \operatorname{curl} \mathbf{F} + \operatorname{curl} \mathbf{G}$

4.6.18. $\operatorname{div}(f\mathbf{F}) = f\operatorname{div} \mathbf{F} + \mathbf{F} \cdot \nabla f$

4.6.19. $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

4.6.20. $\operatorname{div}(\nabla f \times \nabla g) = 0$

4.6.21. $\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl} \mathbf{F} + (\nabla f) \times \mathbf{F}$

4.6.22. $\operatorname{curl}(\operatorname{curl} \mathbf{F}) = \nabla(\operatorname{div} \mathbf{F}) - \Delta \mathbf{F}$

4.6.23. $\Delta(fg) = f\Delta g + g\Delta f + 2(\nabla f \cdot \nabla g)$

C

4.6.24. Prove Theorem 4.17.

4.6.25. Derive the gradient formula in cylindrical coordinates: $\nabla F = \frac{\partial F}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial F}{\partial \theta}\mathbf{e}_\theta + \frac{\partial F}{\partial z}\mathbf{e}_z$

4.6.26. Use $\mathbf{f} = u\nabla v$ in the Divergence Theorem to prove:

$$(a) \text{ Green's first identity: } \iiint_S (u \Delta v + (\nabla u) \cdot (\nabla v)) dV = \iint_\Sigma (u \nabla v) \cdot d\sigma$$

$$(b) \text{ Green's second identity: } \iiint_S (u \Delta v - v \Delta u) dV = \iint_\Sigma (u \nabla v - v \nabla u) \cdot d\sigma$$

4.6.27. Suppose that $\Delta u = 0$ (i.e. u is *harmonic*) over \mathbb{R}^3 . Define the *normal derivative* $\frac{\partial u}{\partial n}$ of u over a closed surface Σ with outward unit normal vector \mathbf{n} by $\frac{\partial u}{\partial n} = D_n u = \mathbf{n} \cdot \nabla u$. Show that $\iint_\Sigma \frac{\partial u}{\partial n} d\sigma = 0$. (Hint: Use Green's second identity.)

This page titled [4.E: Line and Surface Integrals \(Exercises\)](#) is shared under a [GNU Free Documentation License 1.3](#) license and was authored, remixed, and/or curated by [Michael Corral](#) via [source content](#) that was edited to the style and standards of the LibreTexts platform; a detailed edit history is available upon request.

- [4.E: Line and Surface Integrals \(Exercises\)](#) has no license indicated.

☐ Toggle block-level attributions